Mathematical Understandings of Africa

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Africa is a land of many stories, new and old, told in countless languages. Not least among those languages is the one in which, according to Galileo, the book of nature is written: mathematics. Today is a remarkable time in Africa for mathematics and for the stories it can help to tell.

Little noticed by the world, an ecological disaster unfolded on the westernmost coast of Africa in 2003. The Senegal River is one of the major waterways of West Africa, flowing westward from the interior of the continent and forming the northern border of Senegal and the southern border of Mauritania, before veering sharply southward and closely paralleling the Atlantic coast, from which it is separated by a narrow spit of sandy land known as the Langue de Barbarie. Here this mighty river, comparable in scale to the Potomac at Washington or the Hudson at New York, formed a remarkable peninsula over 30 km long with a typical width of less than 300 m. From the point where it narrows, the Langue de Barbarie belongs to Senegal, and there, covering the small Island of N'Dar on the riparian side, lies St. Louis, the former capital of French West Africa and Senegal's second city today. Because of its two distinct fisheries, one oceanic and one freshwater, and because the narrow sandy peninsula afforded an easy land shortcut avoiding a day's upriver travel, St. Louis has long been a regionally important trade and transportation center. With its history and the African cultural events taking place there, against a decaying but still beautiful French colonial backdrop, St. Louis was designated as a World Heritage Site by UNESCO in 2000.

In late 2003 heavy rains caused severe flooding in St. Louis and associated communities. To relieve the flooding, a 4 meter wide ditch was dug across the Langue de Barbarie a few km. south of St. Louis, draining the high waters to the sea. This had the desired effect, but as the citizens of St. Louis watched in dismay over the next few days, the ditch swiftly enlarged to a width of hundreds and, later, thousands of meters, forming a new mouth of the river in a location shifted by more than 20 km. In the blink of an eye, St. Louis changed from a river city to a city of tides and brackish water. This affected the ecology and the way of life of the region, as well as the physical conditions of the dams, bridges, and buildings along the river.

There is no putting Humpty Dumpty back together again. The people of the lower Senegal River must and will adjust. Here is where mathematics touches their lives. The essence of applied mathematics is modeling, the process where a physical reality is replaced by a set of equations that describe what can be quantified and capture how those quantities affect one another. Think of modeling as an observational tool, akin to a microscope, a telescope, or an MRI device. It allows precise visualization and provides a framework for thinking about the thing being observed, and about its potential futures. Applied mathematicians working in

Senegal and elsewhere have taken up the project of modeling the Senegal River basin with the tools of the differential calculus, in terms of flow rates, salinity, geographic features, connectedness and blockages, in an effort to see a certain distance into the future and to engage in rational planning for the economy of the region in order to improve the lives of its population. This is an ongoing project, which will be continuously refined and adapted for the foreseeable future, as the mouth of the river continues to erode and move, as the ecology changes, and as brackish water affects dams and bridges that were not designed to resist salt.

Similar stories are told across Subsaharan Africa. Whenever I have asked Africans what they hope for from mathematics, they reply that they wish to model and predict some pressing environmental, health, or social phenomenon. Again and again the issues Africans most want to understand quantitatively are hydrology, disease, environmental degradation, and economic development. Before any rational response to these problems can be crafted, they have to be understood with quantitative clarity. For data, mathematical modeling is the observational tool of choice.

Africans have recognized this and across the continent have recently embraced mathematics as a key to development. In 2001 Neil Turok, the Director of Canada's Perimeter Institute and a native of South Africa, had the inspiration that Africa was fertile ground for research in applied mathematics. The African population is young and mass education is expanding. Why, he wondered, shouldn't the next Einstein be found among the new generation of Africans? Since mathematics is an inexpensive science, it is a relatively easy matter for a developing country to attain a world level of activity, and the bang for the buck for economic and societal benefits is unparalleled. Africa's human capacity has enormous potential for contributing to the mathematical sciences, and an investment in mathematics has much to return to the continent. Turok made a convincing case in South Africa and founded the African Institute for Mathematical Sciences (AIMS) in Cape Town, which opened its doors in 2003.

With the success of AIMS, Turok thought of replicating it in other countries. His dream of the "Next Einstein Initiative" took a leap forward when he was selected to present it at the 2008 TED Conference. Shortly afterwards a pilot AIMS project was opened in Abuja, Nigeria, within the African University of Science and Technology,. In 2011 a self-sanding institute in mathematics opened its doors in Mbour, Senegal, near Dakar, followed not long afterwards by another AIMS branch in Biriwa, Ghana. These institutions have attracted strong support and participation by prominent European and American mathematicians, including Fields Medalists. This is particularly the case for AIMS Senegal, where an applied mathematician, Mary Teuw Niane, is currently serving as the Minister of Higher Education and where there is a national push to build out high-speed data services and other supporting infrastructure. The AIMS branches offer graduate degrees and specialized training in applied mathematics; for instance in 2014 AIMS Senegal operated a program on modeling fisheries.

There are several ways that mathematics can bring clarity to a problem. Descartes famously taught that all of the geometry of the world we live in is describable by three measurements, distances along a set of perpendicular axes. His insight that data can be understood visually, as geometric relationships of coordinates corresponding to dimensions, forever changed science and technology. Following Descartes, mathematicians characterize complexity in terms of dimensions, which are by no means limited to three. We could of course ask for the three spatial measurements, the x-, y-, and z- coordinates, but in a model of a river we will also ask for flow rates, temperature, salinity, and so on. Each of these, in mathematical parlance, is a new dimension. The complexity of a model is gauged by its dimension, which is a concept to be found in many branches of mathematics. The dimension can be understood as the number of distinct questions we are required to answer in order to understand the model.

As Einstein once quipped, a theory should be as simple as possible, but not more so. The art of mathematical modeling consists in stripping away and ignoring accidental and irrelevant features, leaving the essence, as in philosophy. The fewer the quantities in the model, the better, so long as those that remain are sufficient to understand the object of study. The lower the dimensionality, the greater the clarity. Yet when trying to understand a large amount of data, it may not be at all clear how it can be best organized into an understandable narrative with the fewest dimensions.

Modeling can draw from many branches of mathematics: calculus, statistics, and so forth. However, one of particular interest dates from the work of Joseph Fourier. Although like so many later scientists he was anticipated in certain essentials by Euler, Fourier is credited with a method of understanding data which is to the graphical drawings of Descartes as hearing is to seeing. Just as in daily life some information and patterns are better taken in visually while others are are more readily discerned by ear, so it is with data. Fourier showed that any aural signal can be regarded as the superposition of pure vibrations, and that there are efficient formulae for taking the signal apart and putting it together. He enabled mathematics to accomplish what the brain and ear do when extracting the aspects of our sonic environment which we perceive as music, speech, or indeed any sound we recognize as meaningful in some way.

Today's terms for the pure vibrational tones, as in music, and for the physical forms of those vibrations are the rather unmelodious words "eigenvalues" and "eigenvectors," while the understanding of these concepts is more colorfully termed "spectral analysis." ("Fourier analysis" is a special case.) It is important to note that spectral analysis is computationally easy, fast and cheap. It would be hard to overstate the importance of our second sense organ for data in the development of our technological society, without which the hard sciences and engineering would scarcely have advanced since the Nineteenth Century. Just to cite one example familiar today, when Google performs a search for websites containing some search terms, it carries out a spectral analysis of the part of the internet where those terms appear, finds the dominant eigenvalue, as if that part of the internet were vibrating at that frequency, and presents the searcher with the corresponding eigenvector. A site appears in the list of search results in order of the size of the part of the eigenvector vibrating at that site. In essence, Google finds the loudest musical note made by the network with those search terms, and presents the sites that are playing it the loudest.

"Sonification" is a term being used today for a creative way of determining whether a large data set contains information of the sort we humans can understand and use. Researchers first convert data into sound within our hearing range, using Fourier's methods. Then they listen. Although there are many different ways to thus sonify the data, in practice it does not seem to matter much which one is used. Randomly generated data with no simple patterns will be perceived as noise, perhaps with pops and whooshes but no special tones or rhythmic features. Conversely, if the data are organized, the listener will quickly discern special features that resonate in the mind.

Sonification has become popular enough to sustain an organization devoted to the method, ICAD, the International Community for Auditory Display. Beyond helping researchers identify data sets that contain useful information, sonification enables mathematics to touch art. It has become popular to compose music using scientific data sets, from the motion of molecules to the structure of the galaxy. In Atlanta alone, where Science Taverns and Science Festivals are flourishing, there have within the last year been at least three public performances of music based on data, including "First Life," a mixed-media piece for string quartet composed by Steve

Everett, based on chemical data from Martha Grover's research group at Georgia Tech.

Can mathematics be of comparable service in illuminating culture or artwork? The Luba people of Congo have a story-telling tradition using devices called lukasa memory boards. These are hand-held wooden objects with markings and pegs of beads or shells in special patterns, which the story-teller can look at or touch, thereby calling up information that is germane to the story, somewhat as the modern academic will look at her Powerpoint slide to organize and recall the information she is imparting, perhaps selecting different bits on different occasions for various audiences.

Exactly how the lukasa board can be read and what information is encoded by the markings and pegs is not fully understood. The art of using lukasa boards was traditionally taught only to initiates in a secret society known as bambudye, whose number has greatly dwindled since the height of the Luba culture in the 18th and 19th Centuries. It is believed that the stories in the lukasa boards include genealogies, political relations, geography, and history, but scholars are hazy about the precise content and how it is encoded. Mathematics can no more bring back a lost narrative art than it can restore the past mouth of a river, but it can be of assistance by identifying patterns and determining how much information is encoded in them. Many lukasa boards are preserved in museums and collections, so data are available. Speculatively, the data probably come in two mathematical types: geometric, and relational. Different cultures understand spatial relationships in different ways, so if lukasa boards contain geographic or other geometric information, the way the information is represented may not be at all in the form of a map, as understood in the Western tradition, with Cartesian or other coordinates. Nonetheless, modern geometry includes a highly developed theory of the different ways a given spatial concept can be represented, of the transformations changing one representation into another, and of the aspects of the geometry that are invariant and independent of the way it is represented. With data from enough examples of lukasa boards, it may be possible to recognize which features recur and have the characteristics of spatial information. If so, one may be able to transform that information into a set of maps with features that could be compared with extant drawings and maps which we understand in our own cultural tradition. Perhaps somewhere there is a match that could serve as a Rosetta Stone for deciphering the lukasa boards!

Reports that lukasa boards contain genealogies and political relations point to another branch of mathematics, the theory of networks, or graphs†. In this theory, all information is stripped away except for information concerning which entities are connected to which. As a common example, one could consider a social group organized by a single relationship, such as marriage or friendship on social media. The members of the group are represented as simple dots ("vertices"), which are connected by links ("edges") when they have the relationship we specify. All additional information is disregarded. Mathematics brings few new insights when

† In the mathematical community the more common term is "graph," but the reader should be aware that in this context the word has nothing to do with the more widely familiar Cartesian notion of the graph of a function.

the data set, or number of vertices, is small, but when the data set is large, it can reveal many important and even surprising insights, the Google algorithm mentioned above among many other examples.

Let us suppose that by examining the features of lukasa boards, it is possible to say that in a given board there are so many entities (people? villages? dynasties?) and so many relationships, and to organize these relationships as graphs. One could ask how many different questions the bambudye story-teller is prepared to answer about those entities and their relationships according to the lukasa board: e.g., who is related in a certain way to whom, which events happened in which reign, which villages are allied for purposes of trade, kinship, or war. While this level of mathematical abstraction will cast no light on what the relationships are, it may be possible to identify how many organizing principles there are, quantified as a number of dimensions. That is, how many questions are needed in order to understand the story – if only we knew what those questions were!

Because of the glut of data facing the world today, researchers are keen to find new mathematical tools for extracting information from complexity. This author, with Joachim Stubbe, has been among those expanding the toolbox, by making connections between spectral analysis and dimensionality. In a recent article we show how the dimension that characterizes a graph can be read off from the spectrum of its eigenvalues. The idea of using this tool, along with the more established ones of statistical and geometric analysis, to seek the social information encoded in lukasa boards is intriguing. It would be as if by listening to the musical sounds of a story told in an unknown language one could discern at least the number of characters and the complexity of their relationships.